Mathematische Assistenzsysteme

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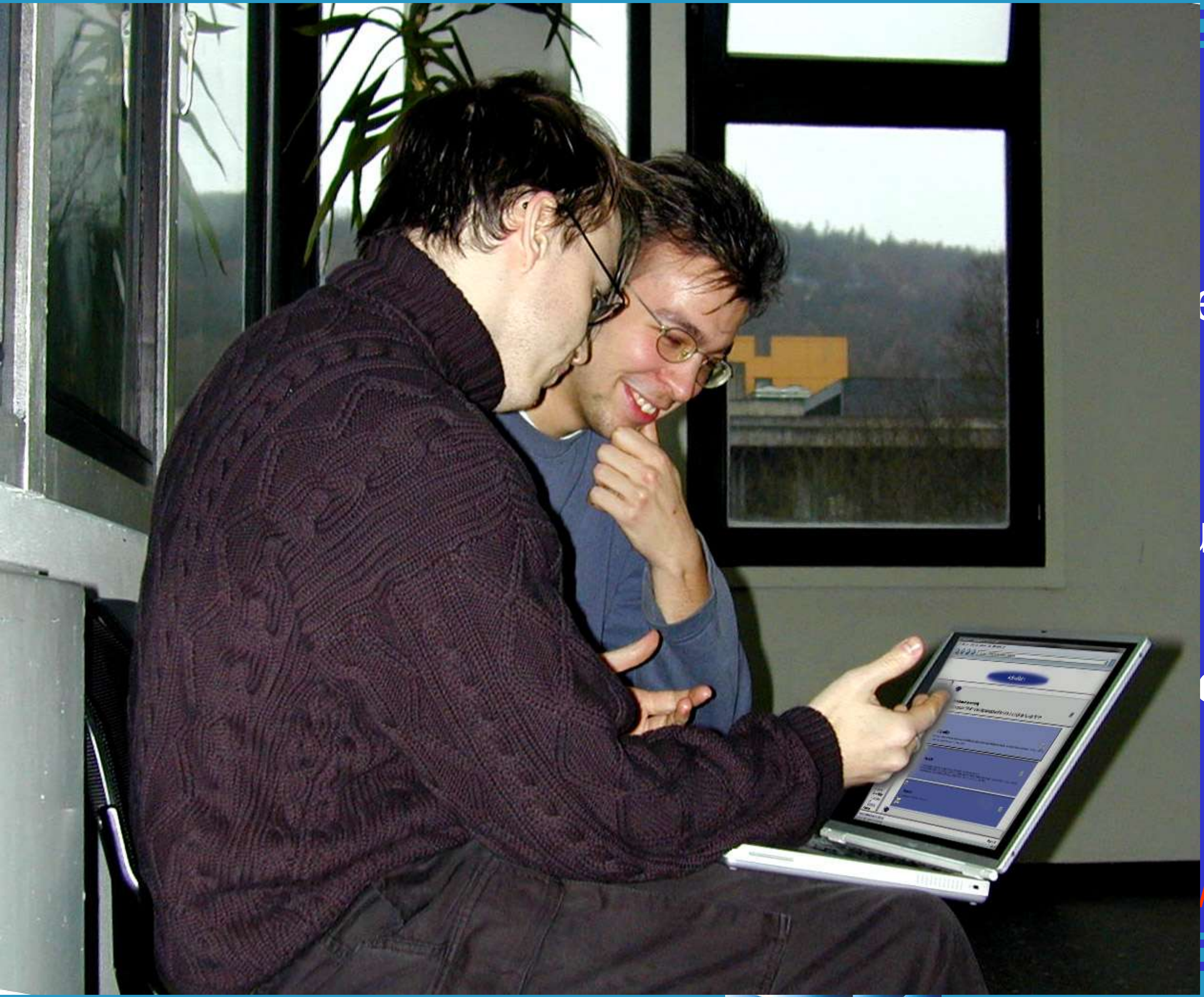
Was ist ein mathematisches Assistenzsystem?



en Datenbank

und Lemmata

eme

w.activemath.org

..... to set the stage.

1954: Martin Davis

Theorem: The Sum of two even numbers is again even
Proof: (Presburger Arithmetic)

1956: Alan Newell, Herb Simon

Theorems: from Principia Mathematica
Proof: Logic Theorist

Dartmouth Conference

Scruffy — Psychology

Logic Theorist
GPS
Newell
Woody Bledsoe

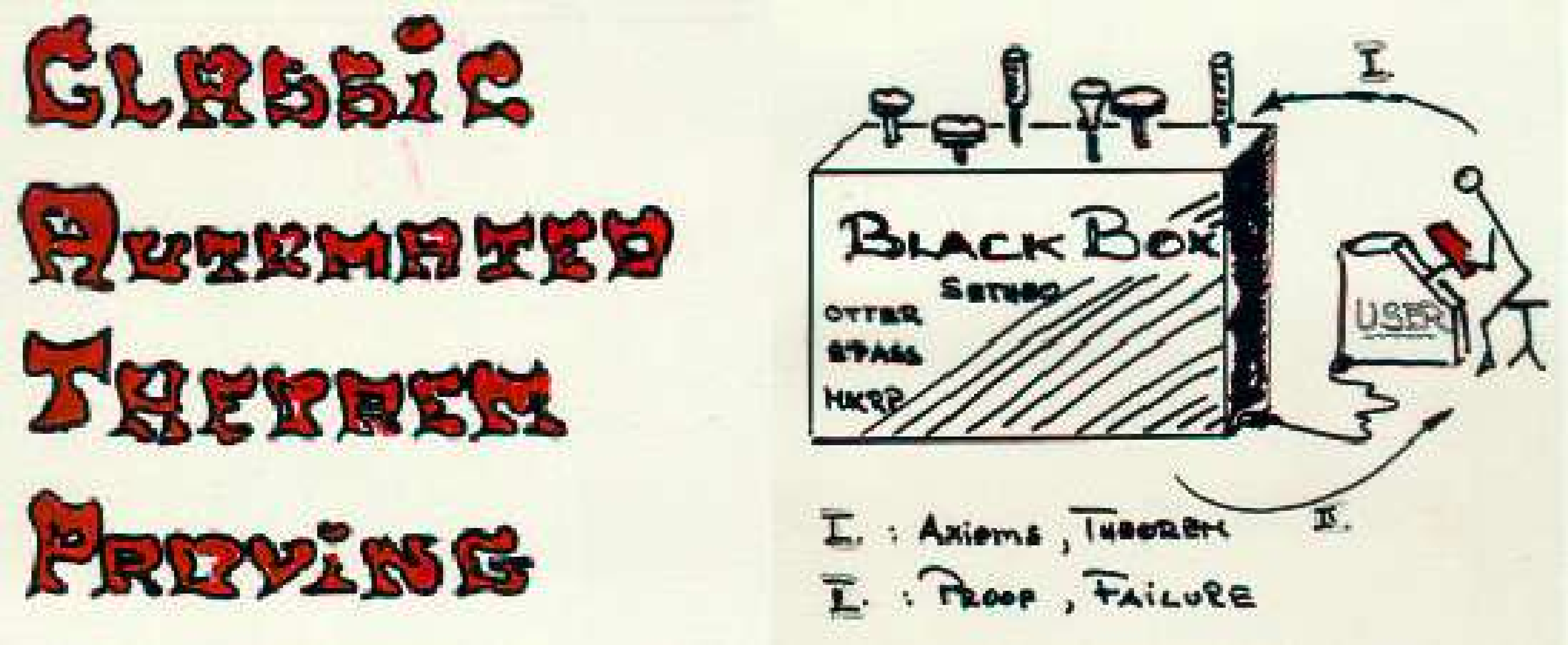
Neat — Logic

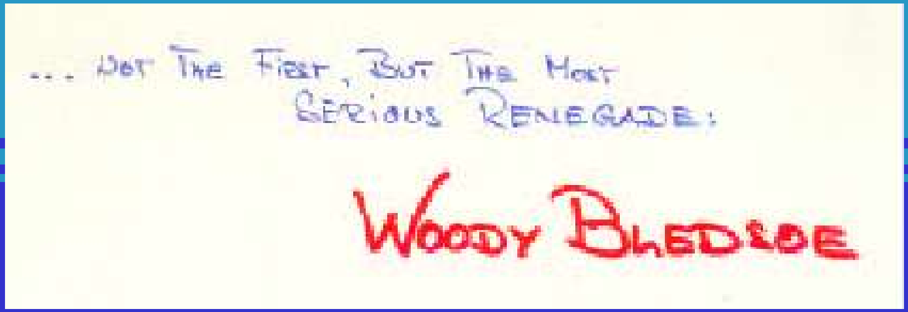
Matrix
Wang
Resolution

3 Paradigms:

1. Classical Automated Theorem Proving
   - Resolution
   - Tableaux-Methods
   - Matrix and Connection Method
2. Tactical Theorem Proving
   - Automath
   - NUPRL
   - IMPS
   - ISABELLE etc.
3. Human oriented Theorem Proving
   - Natural Deduction
   - Woody Bledsoe
   - Proof Planing: OY$_S$^{T}ER-CLAM, MEGA

Deduction Systems





Automated theorem proving is not the beautiful process we know as mathematics.
This is „cover your eyes with blinders and hunt through a cornfield for a diamond-shaped grain of corn“ ...

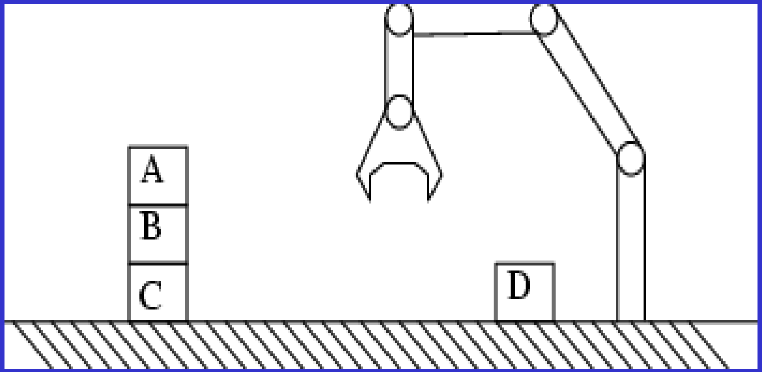
Mathematicians have given us a great deal of direction over the last two or three millennia.
Let us pay attention to it.

Woody Bledsoe, 1986

Can we do better ?

Knowledge based Proof Planning

AI-PLANNING IN THE BLOCKS WORLD



- Initial State

  on(A,B), on(B,C), on_table(C),
  on_table(D), free(A), ...

- Goal

  on_table(B)

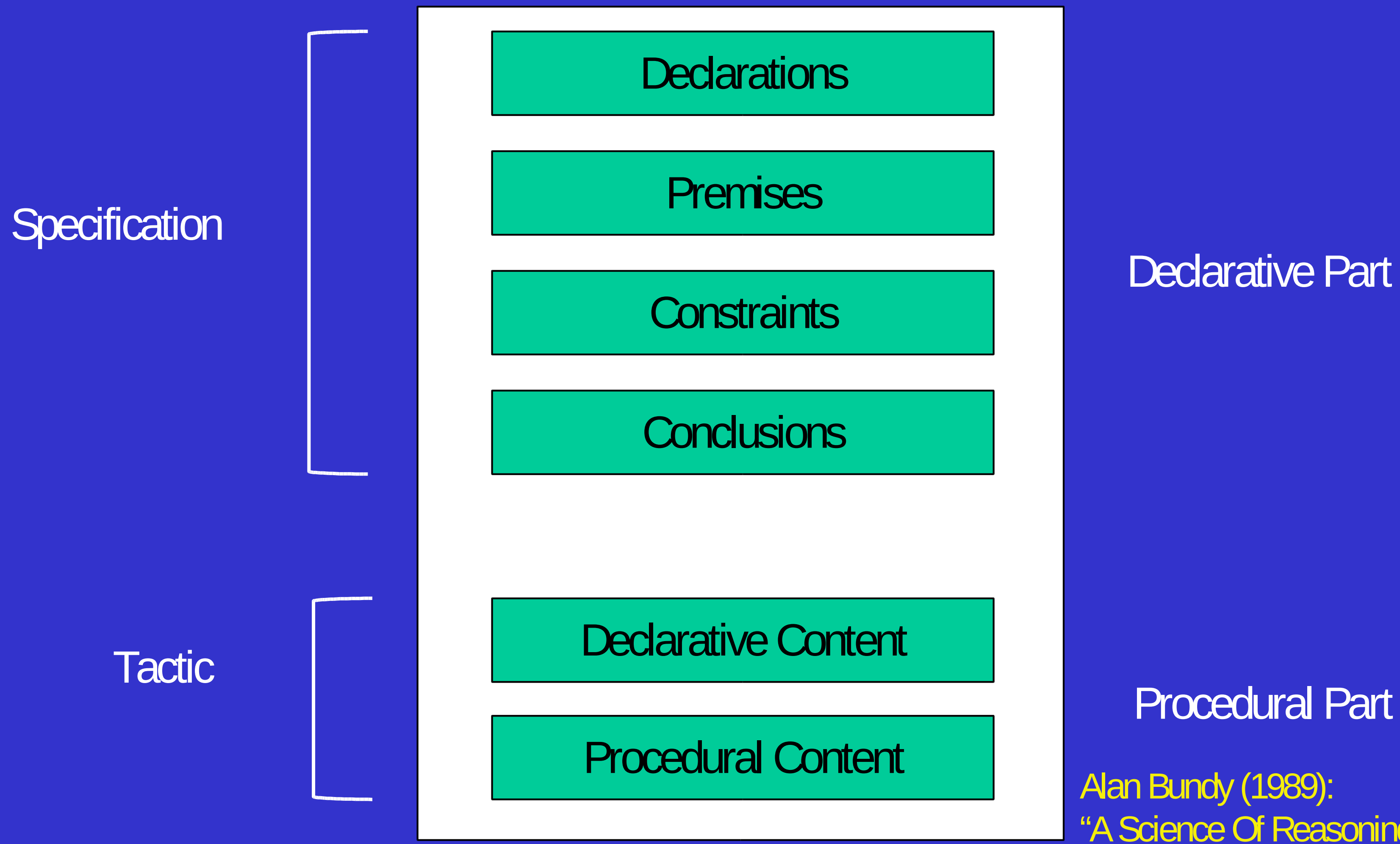
- Operators

  PUTDOWN(X):
  precondition: holding (X)
  effect: (+) on_table(X), hand_empty
  (-) holding(X)

- Plan

  pick(A), putdown(A), pick(B), putdown(B)

Methods in Proof Planning

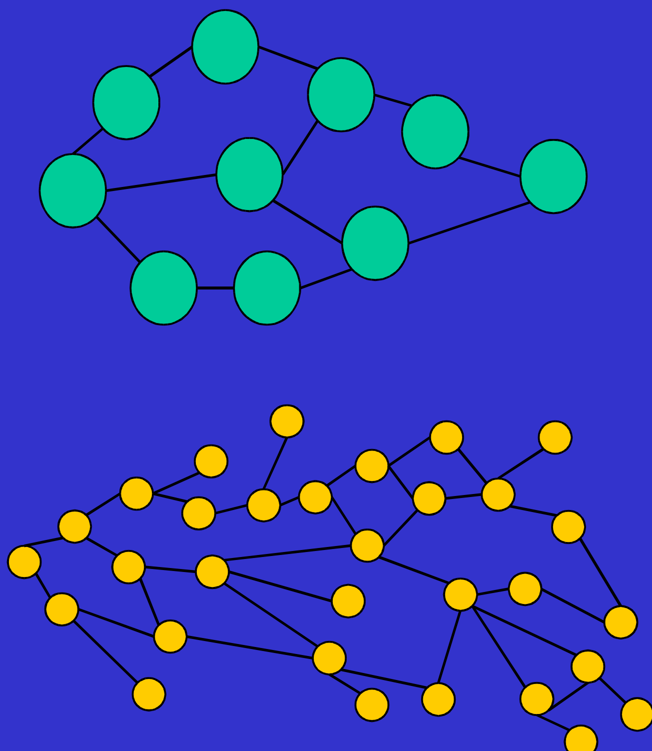


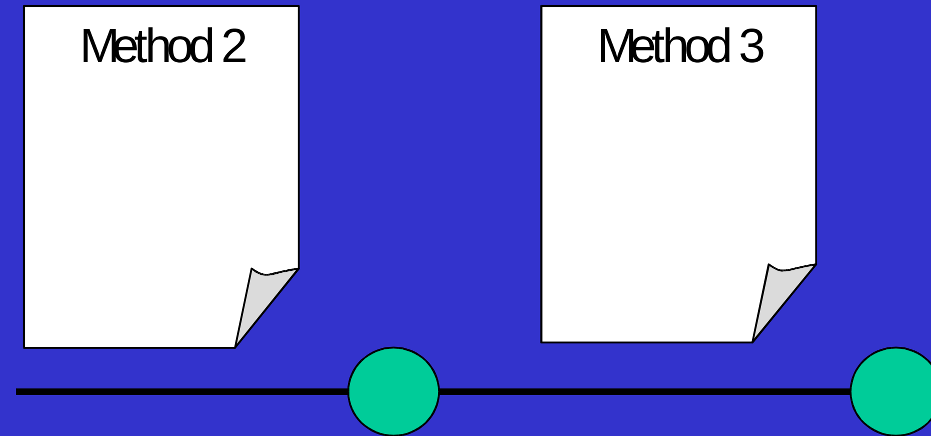
Alan Bundy (1989):
“A Science Of Reasoning“

Methods: An Example

method: `Indirect`				
premises	$\oplus$L2			
conclusions	$\ominus$L4			
appl.cond				
proof schema	L1. $\neg Th$	$\vdash$	$\neg Th$	(HYP)
	L2. $\Delta\ \neg Th$	$\vdash$	$\perp$	(OPEN)
	L3. Δ	$\vdash$	$\neg\neg Th$	($\neg$I;2)
	L4. Δ	$\vdash$	Th	($\neg$E;3)

Knowledge based Proof Planning





- Classical Proof Planning
- Knowledge based Proof Planning

Ressources
- Knowledge
- Time
- User Interaction

Mathematical Control Knowledge

Global mathematical control:

- Prove $|a| < b$ directly or via auxiliary variabels
  $\Rightarrow$ prove $|a| < b$ by **`Solve<b`**, **`Solve*`** or
  ... **`LimHeuristic`**.

- Use important parts of assumptions to introduce
  auxiliary variabels/inequalities:
  e.g. LimHeuristic requires:
  - Focus
  - UNWRAPHYP
  - REmoveFocus
  - MP-b

Source: Erica Melis

Control knowledge represented as rules:

```
(control-rule attack-inequality
    (IF (goal-matches (?goal (?x < ?y))))
    (THEN
      (prefer((Solve< ?goal)
             (Solve* ?goal)
             (ComplexEstimate ?goal)
             (Simplify ?goal))))
```

```
(control-rule case-analysis-intro
    (IF   (last-method (Rewrite (?C -> ?R))) AND
          (failure-condition (trivial ?C)))
    (THEN (select (CaseSplit (?C or not ?C)))))
```

Source: Erica Melis

Peter Deussen: Semigroups and Automata,

Springer Verlag, 1971

Theorem 4.8: *Let and be two equivalence relations.*

Then (∪) ^{t} is also an equivalence relation.

Proof: (Idea)

To be shown:

- Symmetry
- Reflexivity
- Transitivity

of (∪) ^{t}

No	S;D	⊢ Formula	Reason
1.	1;	$\vdash$ EqRel(σ)	(Hyp)
2.	2;	$\vdash$ EqRel(ρ)	(Hyp)
3.	1;	$\vdash$ ref(σ) $\wedge$ symm(σ) $\wedge$ trans(σ)	(Def-EqRel 1)
4.	2;	$\vdash$ ref(ρ) $\wedge$ symm(ρ) $\wedge$ trans(ρ)	(Def-EqRel 2)
5.	5;	$\vdash \forall\tau.\forall\mu.\forall x.(\tau \cup \mu)(x) \Leftrightarrow (\tau(x) \vee \mu(x))$	(Def-Union)
97.	1,2;5	$\vdash$ ref($(\sigma \cup \rho)^t$)	(PLAN)
98.	1,2;5	$\vdash$ symm($(\sigma \cup \rho)^t$)	(PLAN)
99.	1,2;5	$\vdash$ trans($(\sigma \cup \rho)^t$)	(PLAN)
Thm.	1,2;5	$\vdash$ EqRel($(\sigma \cup \rho)^t$)	(Def-EqRel 97 98 99)

More Examples: epsilon-delta Proofs

- Summensatz (LIM+)

  $$\lim_{x\to a} f(x) = L_1 \wedge \lim_{x\to a} g(x) = L_2 \to \lim_{x\to a} f(x) + g(x) = L_1 + L_2$$

- Produktsatz (LIM*)

  $$\lim_{x\to a} f(x) = L_1 \wedge \lim_{x\to a} g(x) = L_2 \to \lim_{x\to a} f(x) * g(x) = L_1 * L_2$$

- LIM$-$, ContIfDeriv, Continuous+, Continuous$-$, Continuous*, ContCompos, $\lim_{x\to a} x^2 = a^2$ etc.

$\lim_{x\to a} f(x) = L$:

$$\forall\epsilon(0 < \epsilon \to \exists\delta(0 < \delta \wedge \forall x(|x - a| < \delta \wedge x \neq a \to |f(x) - L| < \epsilon)))$$

Woody Bledsoe: “Challenges“

Method for Limit Theorems

method: `ComplexEstimate`	
premises	L1, ⊕L2, ⊕L3, ⊕L4
conclusions	⊖L7
appl.cond	$\exists k, l, \sigma(\texttt{CASextract}(a,b) = (k,l,\sigma))$
proof schema	L1. Δ ⊢ $\lvert a \rvert < e_1$ () L2. Δ ⊢ $\lvert k \rvert \leq M$ (OPEN) L3. ⊢ $\lvert a_\sigma \rvert < \epsilon/2 * M$ (OPEN) L4. Δ ⊢ $\lvert l \rvert < \epsilon/2$ (OPEN) L5. ⊢ $b = b$ (Ax) L6. ⊢ $b = k * a_\sigma + l$ (CAS;L5) L7. Δ ⊢ $\lvert b \rvert < \epsilon$ (fix;L2, L3,L4,L6)

$$\texttt{CASextract}(\underbrace{f(X_1) - l_1}_{a}, \underbrace{f(x) + g(x) - (l_1 + l_2)}_{b}) = (1, (g(x) - l_2), [x/X_1])$$

Source: Erica Melis

Construction of mathematical Objects

CONSTRAINT SOLVING:

Collecting constraints and check for consitency

Final constraint store for LIM+

$$
\begin{array}{ccccl}
0 & < & E_2 & \leq & \epsilon/2; \\
0 & < & D & \leq & \delta_2, \delta_1; \\
0 & < & E_1 & \leq & \epsilon/(2 * \mathbf{M}), \epsilon/2; \\
1 & \leq & \mathbf{M} & < & \epsilon/(2 * E_1); \\
-\infty & < & X_1 = x = X_2 & < & +\infty
\end{array}
$$

Source: Erica Melis

Proof Presentation to the User

Verbalisation of `ComplexEstimate`:

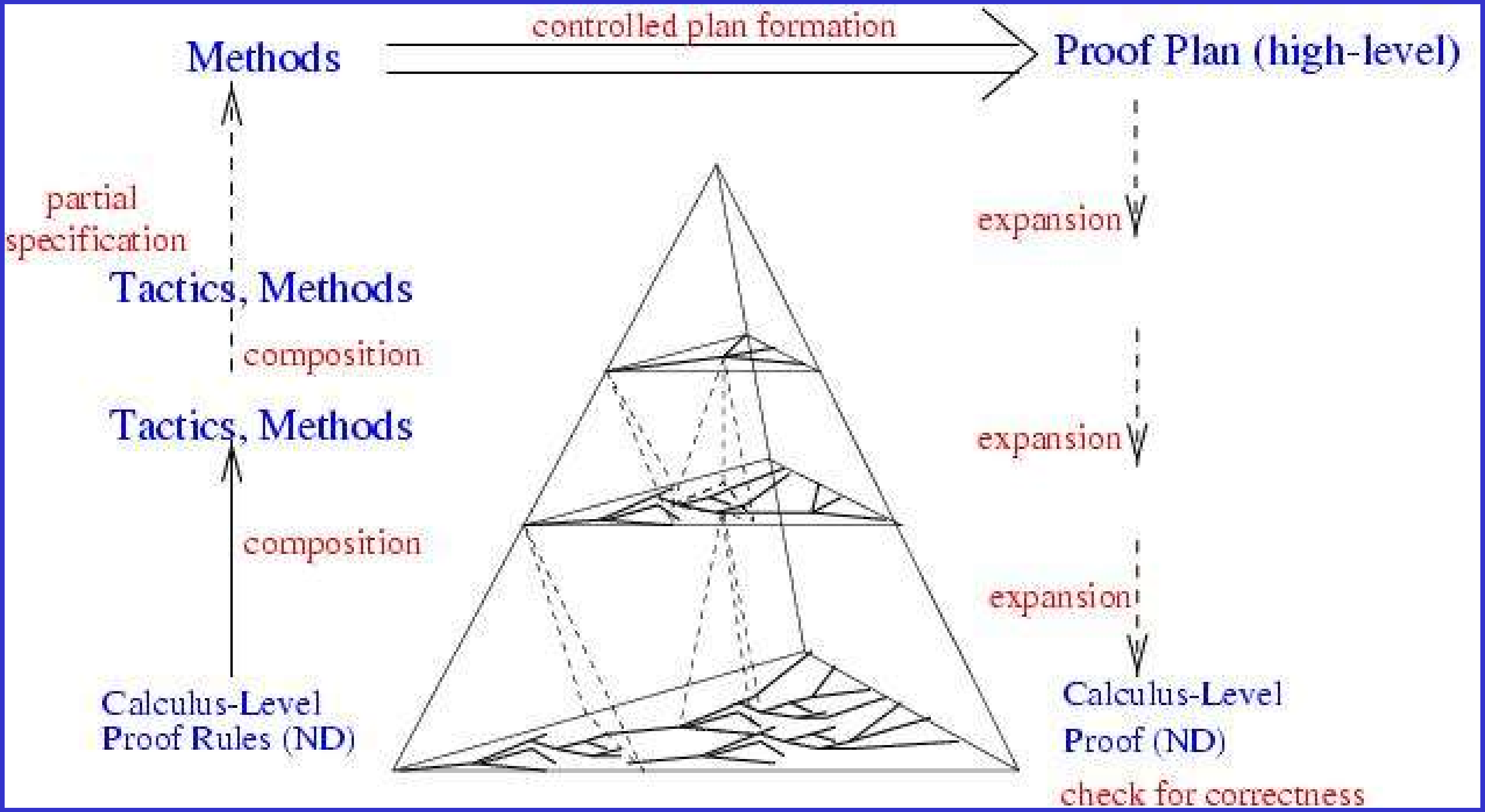
In order to estimate the magnitude of $|b|$
we rewrite the term to $|k * a + l|$
and use the Triangle Inequality $|k * a + l| \leq |k * a| + |l|$.
Now the goal can be shown in three steps:

- There exists an M such that $|k| < M$ and
- $|a| < \epsilon/(2 * M)$, and
- $|l| < \epsilon/2$.

Then $|b| \leq |k| * |a| + |l| < M * \epsilon/(2 * M) + \epsilon/2 = \epsilon$
and therefore $|b| < \epsilon$.

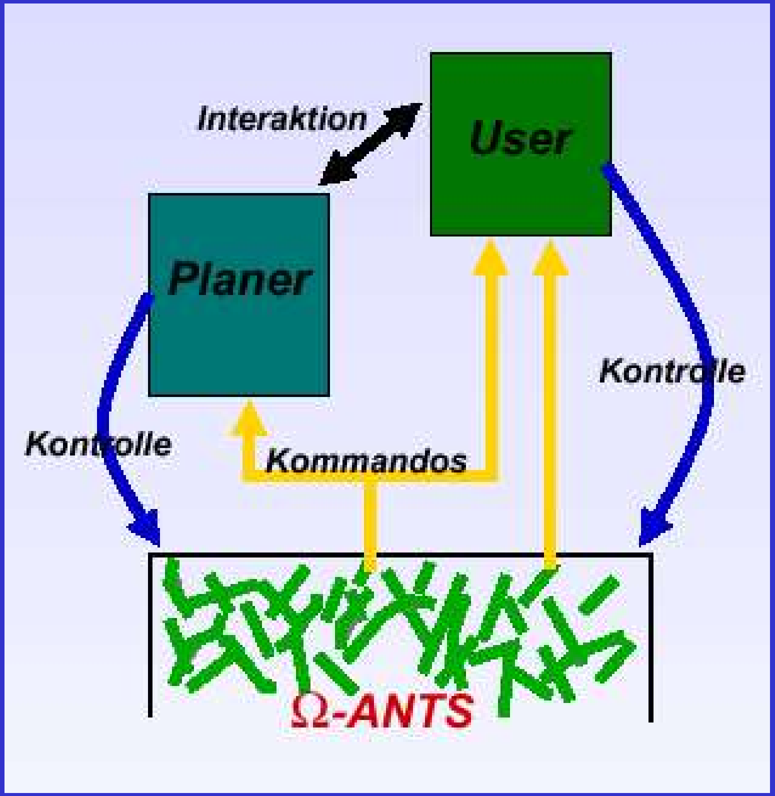
Source: Erica Melis

PDS: Representation of (partial) Proofs



MEGA -ANTS: Combining ATP with Proof Planning

- concurrency and ressource adaptive behaviour
- anytime algorithms
- flexible integration of:
  - natural deduction
  - tactics and methods
  - external systems



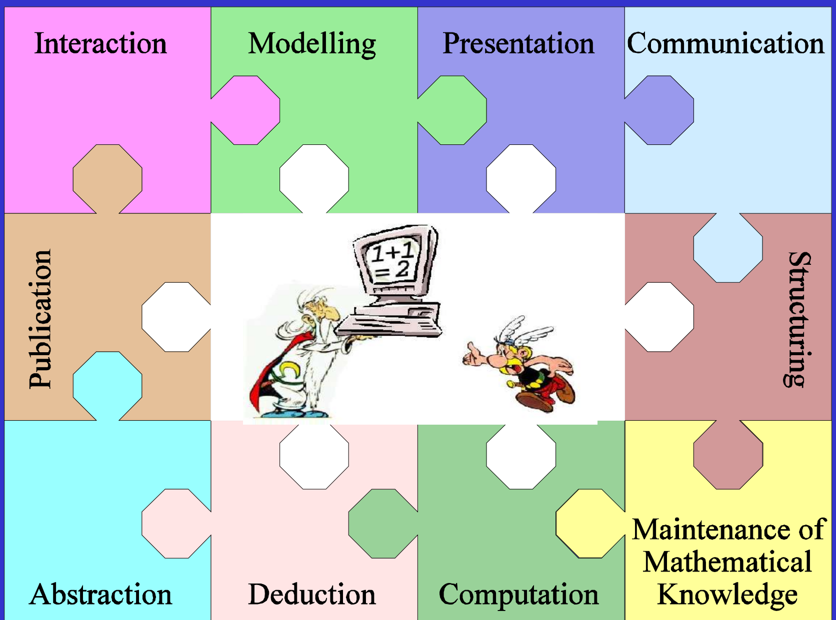
Chris Benzmüller, Volker Sorge

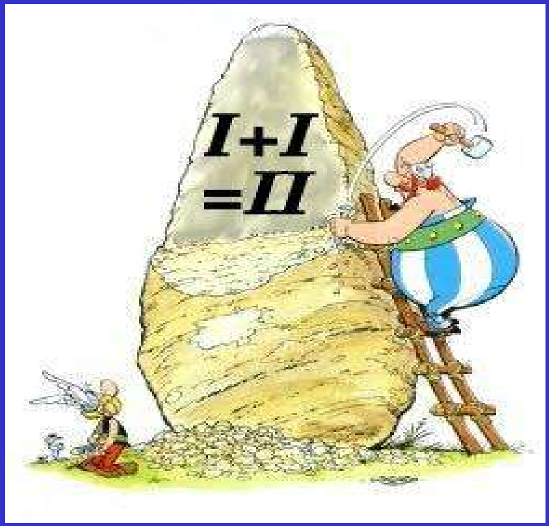
Mathematical Assistance Systems

Integrated Mathematical
Assistant Environment

vs.

'Pen-and-Paper'
Mathematics





<u>Applications</u>

Mathematics reserach

Mathematics education

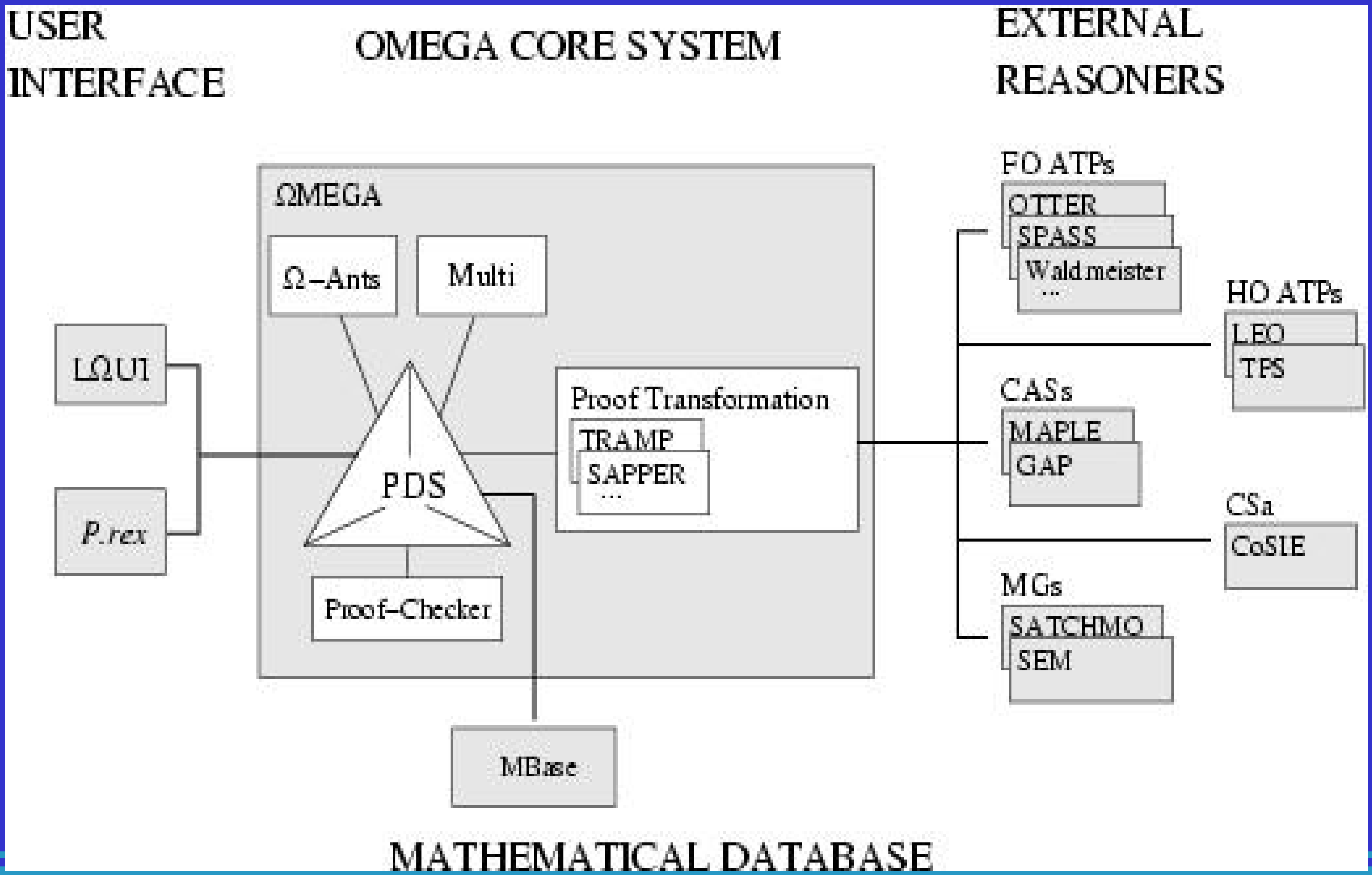
Formal methods

<u>Join of resources necessary</u>

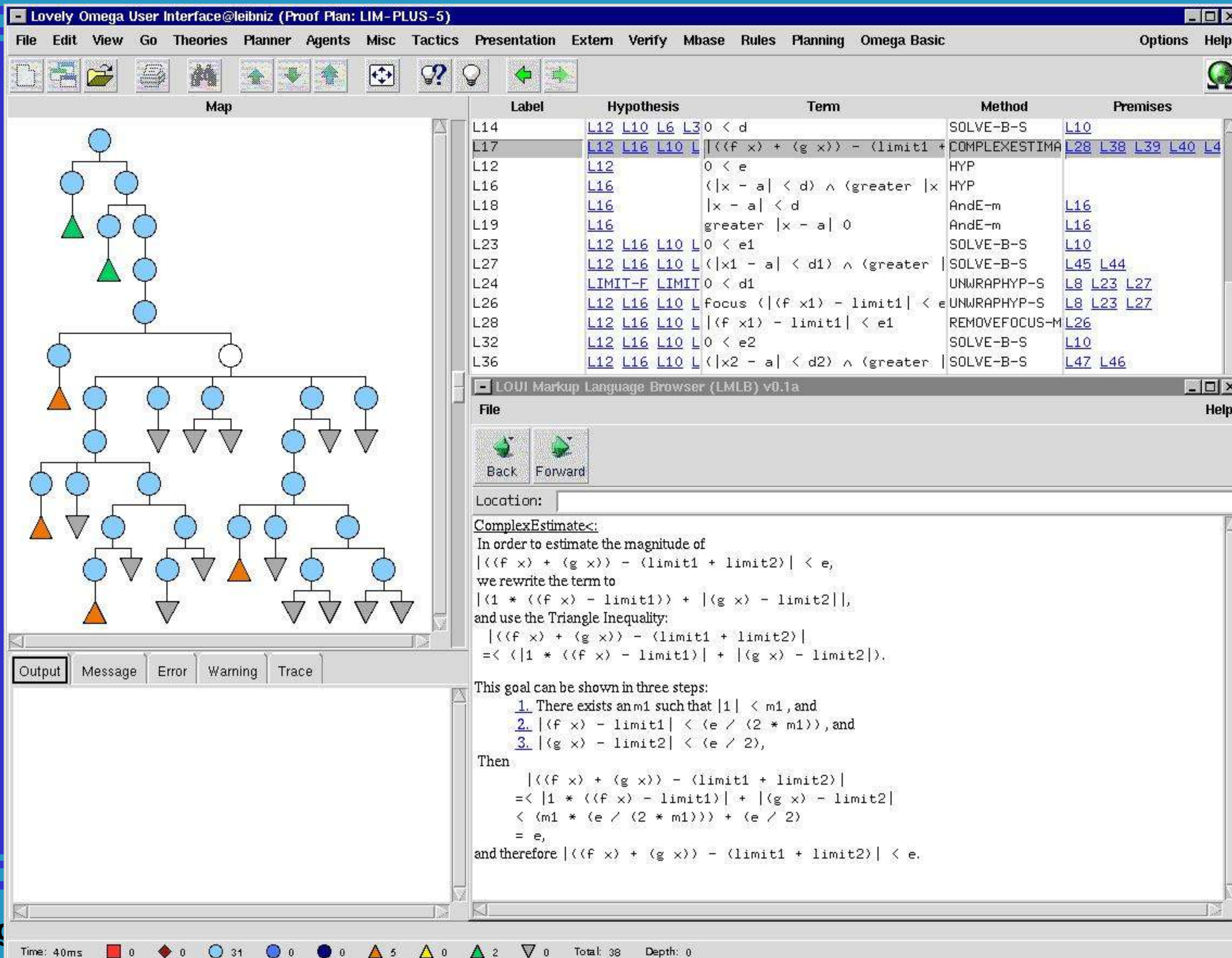
<u>System level:</u> Coq, NuPrl, Isabelle/HOL, PVS, Theorema, ΩMEGA, Clam, ...

<u>Research Networks:</u> Calculemus, MKM, Monet, MoWGLI

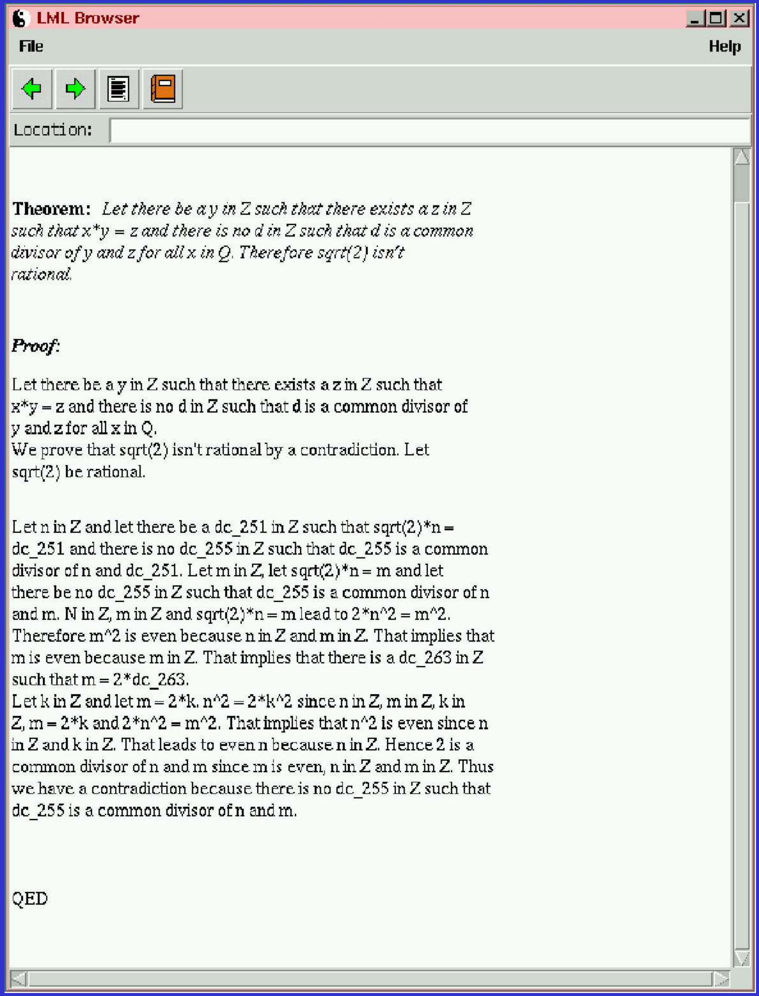
The OMEGA SYSTEM



Proof Planning: A Screen Shot



Proof Verbalization



P.REX (successor of PROVERB):

- lifting of proofs in the PDS to assertion level
- macro-planning text structure
- micro-planning sentence structure and linguistic realization
- generation of natural language representation
- pre-required: linguistic knowledge
- user-adaptive proof explanation

Zwei Entwicklungsrichtungen:

CHALLENGE:

Ein integriertes mathematisches Assistenzsystem

Grundlagenforschung!

Knowledge Representation for Mathematics

– XML-Representation
– Semantics (OpenMath) extended by meta data (publ, mathematical, and pedagogical)

– Formal content for

- Calling external systems
- Intelligent search functionalities

⇨ Mathematical Ontology

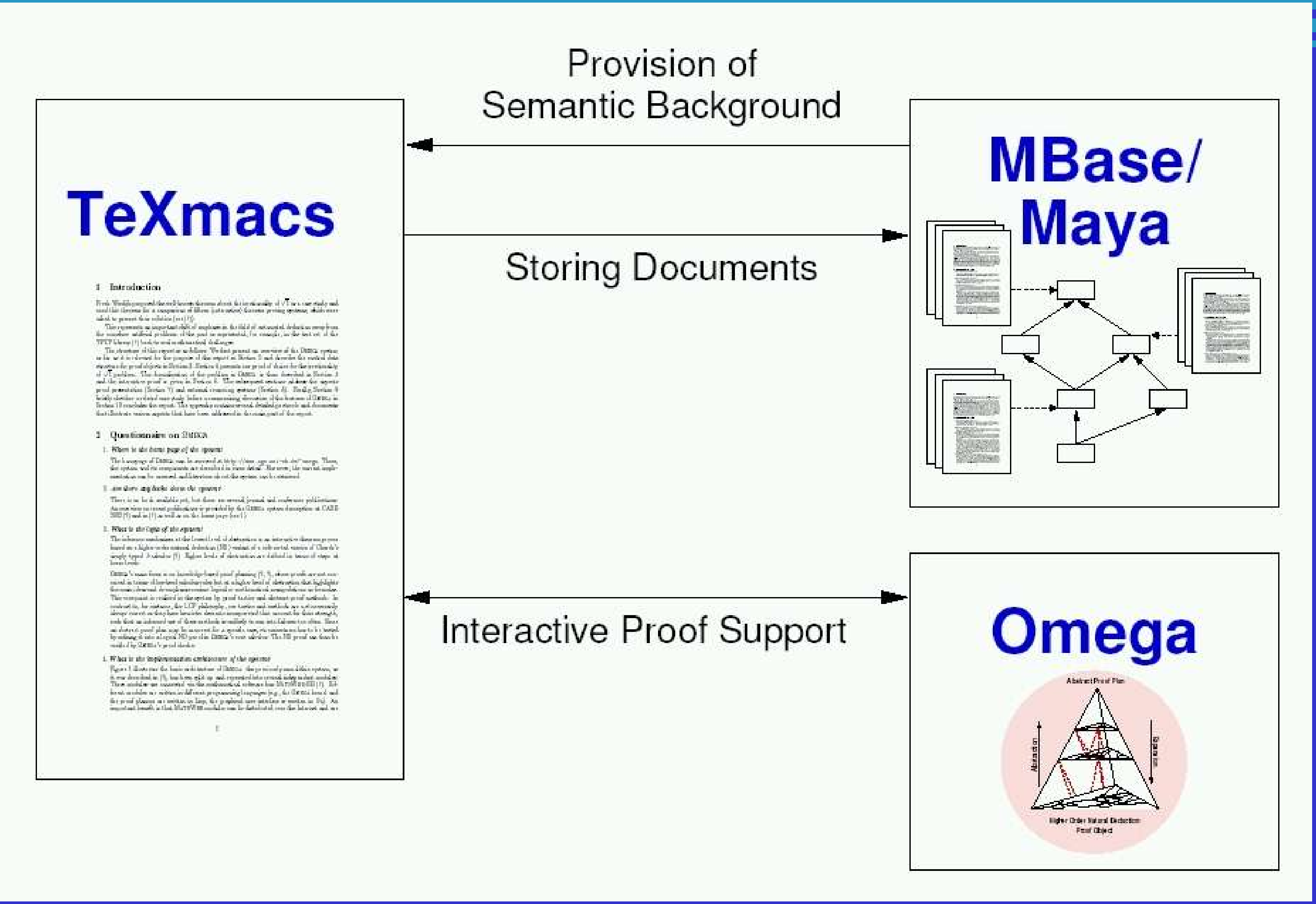
Knowledge: the building blocks in OMDoc

```
<definition id="c6s1p4_Th2_def_monoid" for="c6s1p4_monoid">
<metadata>
    <depends-on>
        <ref theory="cp1_Th3" name="structure" />
    </depends-on>
    <Title xml:lang="en">Definition of a monoid</Title>
</metadata>
<CMP xml:lang="en" format="omtext">
A monoid is a <ref xref="cp1_Th3_def_structure"> structure </ref>
<OMOBJ>
<OMS cd="elementary" name="ordered-triple"/>
    <OMV name="M"/> <OMS cd="cp4_Th2" name="times"/> <OMS cd="cp4_Th2" name="unit"/>
</OMOBJ> in which
<OMOBJ>
<OMS cd="elementary" name="ordered-pair"/>
    <OMV name="M"/> <OMS cd="cp4_Th2" name="times"/>
</OMOBJ> is a semi-group
with <ref xref="c6s1p3_Th2_def_unit">unit</ref>
<OMOBJ xmlns="http://www.openmath.org/OpenMath">
    <OMS cd="cp4_Th2" name="unit"/>
</OMOBJ>.
</CMP>
<FMP><OMOBJ> ... </OMOBJ></FMP>
</definition>
```

An Example:

A MONOID

Ein mathematisches Assistenzsystem



Computer Supported Mathematics !!

Schickard:
Die erste mechanische Rechenmaschine der Welt.

. . . . . . **Zuse:** die erste elektronische Rechenmaschine.