V. Simulation

Sven Johr

Universität des Saarlandes

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Datennetze II/Verifikation II

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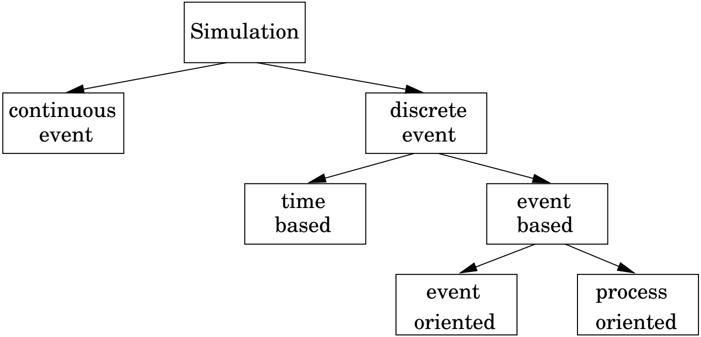
Outline

1. Discrete Event Simulation
   - Introduction
   - Random numbers
2. Statistics
   - Value estimation
   - Hypothesis testing
3. Model checking

Simulation

- Instead of numerically analysing a system.
- Perform single runs of the system.
  - Define a stopping criterion Ψ,
  - continue simulating the system until Ψ is fulfilled.
- Collect information from single runs and make a conclusion.
- No exhaustive simulation,
  - result has some uncertainty.
- Time-advance mechanism is used.
  - Clock times are sampled,
  - simulation clock advances in a discrete step.
  - Simulation time $\neq$ real time.

Classification



- Discrete-event simulation: systems are discrete state systems.
- Time is continuous.

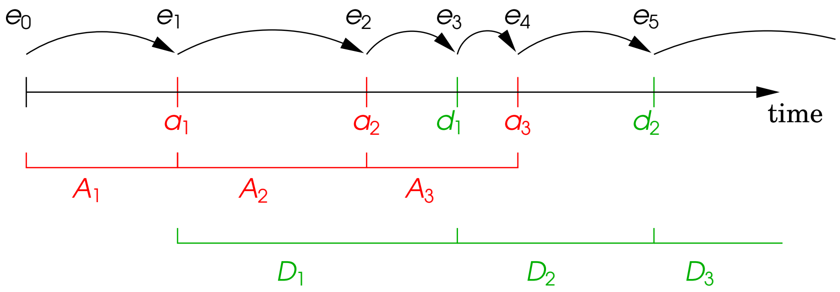
TIME-BASED SIMULATION

- Define fixed step size Δt.
- Check whether events happen in $[t, t + \Delta t]$.
- If so, execute events.
- Advantage.
  - Easy to implement.
- Disadvantages.
  - Events are assumed to have no order,
  - events are assumed to be independent.

Event-based simulation

- Time steps have variable length.
- Occurrence of events controls length of time step.
- Exactly one event per time step.
- Actual event causes future events to occur.
  - Gathered in an ordered event list.
- Events have to be inserted in order.

EXAMPLE – G|G|1 SIMULATION



Implementation strategies

- Event oriented.
  - Procedure P_i for every event-type i.
  - P_i invoked if occurring event is of type i.
- Process oriented.
  - Associate process with each event-type.
  - Processes interchange information via communication.
  - Scheduling of events is done implicitly.

. . . AND PSEUDO RANDOM NUMBERS

- Generate random number from given probability distribution.
- Has to conform to the given distribution, otherwise
  - obtained simulation results are suspicious.
- True random numbers can not be generated with deterministic algorithms.
- Pseudo-random numbers are used.
  - Generate pseudo-random series on finite subset of $\mathbb{N}$.
  - Compute pseudo-uniformly distributed numbers on $[0, 1]$.
  - Verify if generated numbers can be regarded as true random numbers.
  - Compute non-uniform distributed pseudo-random numbers.

SET-UP

- n single runs or samples of the system have been performed.
- n is called the sample size.
- Measures of interest are recorded for each sample, e. g.,
  - (expected) waiting time in a queue,
  - (expected) number of jobs in a queue.
  - Remark: for both examples one has to simulate more than one customer!
- Statistics is used to estimate measure of interest for the complete system, i. e., for all possible system runs.

Obtaining a concrete value

- E. g., estimation of mean value.
- Assert that mean value lies in a particular interval with given certainty.
- Interval is called confidence interval.
- Certainty is called confidence level.

Estimating the mean

- Estimate unknown mean value μ of random variable X.
- X is supposed to have unknown variance σ^2.
- Our simulation generate n samples $x_i, i = 1, 2, \ldots, n$.
- Each x_i is a realisation of random variable X_i.
- $X_i, i = 1, 2, \ldots, n$ are independent and identically distributed with
  - $E[X_i] = \mu$,
  - $\sigma^2_{X_i} = \sigma^2$.

Estimator

- Estimator $\bar{X}(n)$ should be
  - unbiased, i. e., $E[\bar{X}(n)] = \mu$.
  - Intuition.
    1. Perform very large number of experiments,
    2. each resulting in an estimator $\bar{X}_i(n)$,
    3. average of $\bar{X}_i(n)$ will be μ.
- Point estimator for the sample mean is $\bar{X}(n) = \frac{\sum_{i=1}^{n} X_i}{n}$.
- Point estimator for the sample variance is $S^2(n) = \frac{\sum_{i=1}^{n} [X_i - \bar{X}(n)]^2}{n-1}$.

Estimating the mean

- Problem with $\bar{X}(n)$ is
  - how close is it to μ?
  - On one experiment it may be close,
  - on another it may differ by a large amount.
- $\bar{X}(n)$ is a random variable with variance
  - $\mathrm{Var}[\bar{X}(n)] = \frac{\sigma^2}{n}$.
- An unbiased estimator of the variance is
  - $\widehat{\mathrm{Var}}[\bar{X}(n)] = \frac{S^2(n)}{n}$.

Confidence interval - 1

- Construct the random variable $Z_n = \frac{\bar{X}(n)-\mu}{\sqrt{\sigma^2/n}}$.
- Define $F_n(z) = \mathbf{Pr}(Z_n \leq z)$, i. e., $F_n(z)$ is the probability distribution function of Z_n.

Theorem (Central limit theorem)

$F_n(z) \to \Phi(z)$ as $n \to \infty$, with

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

- $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$,
  - density function of normal distribution with mean μ and variance σ^2.

Confidence interval - 2

- Intuition behind Central Limit Theorem.
  - Random variable $Z_n(z)$ is for large n distributed as a standard normal random variable
  - independent of the distribution of the X_i.
- Thus, $\bar{X}(n)$ is distributed as a normal random variable with mean μ and variance $\frac{\sigma^2}{n}$.
- Generally, σ^2 is unknown.
- Replace σ^2 by $S^2(n)$ for sufficiently large n.
- $t_n = \frac{\bar{X}(n)-\mu}{\sqrt{S^2(n)/n}}$ is approximately distributed as a standard normal random variable.

CONFIDENCE INTERVAL - 3

- For large n it follows

$$\begin{aligned} & \mathbf{Pr}\left(-z_{1-\alpha/2} \leq \frac{\bar{X}(n)-\mu}{\sqrt{S^2(n)/n}} \leq z_{1-\alpha/2}\right) \\ = & \mathbf{Pr}\left(\bar{X}(n) - z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}} \leq \mu \leq \bar{X}(n) + z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}}\right) \\ \approx & 1-\alpha \end{aligned}$$

- Where $z_{1-\alpha/2}$ denotes the $1-\alpha/2$ single sided critical value of the standard normal distribution.

Confidence interval - 4

- For sufficiently large n a confidence interval with confidence level $1 - \alpha$ is given by

$$\bar{X}(n) \pm z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}}$$

- Intuition.
  1. Let $\beta = 1 - \alpha$ be the desired confidence level,
  2. construct a large number of independent confidence intervals with confidence level β,
     - each based on n observations, with sufficiently large n,
  3. the proportion of intervals containing μ is β.

Confidence interval - 5

- What does it mean?

  ... n sufficiently large...

  - Too small n will cause a confidence level less than $1 - \alpha$.
- t_n is called a Student's t distribution with degree of freedom $n - 1$.
  - For $n \to \infty$, t_n approaches the normal distribution.
- For a Student's distribution with $n - 1$ degrees of freedom $t_{n-1,1-\alpha/2}$ is the $1 - \alpha/2$ one sided critical value.

EXAMPLE

- $(x_1, x_2, \ldots, x_5) = (0.108, 0.112, 0.111, 0.115, 0.098)$.
- Sample mean $\bar{X}(5) = \frac{\sum_{i=1}^{5} x_i}{5} = 0.1088$.
- Sample variance $S^2(5) = \frac{\sum_{i=1}^{5}(x_i - \bar{X}(5))^2}{5-1} = 0.0000427$.
- Confidence level is supposed to be $1 - \alpha = 0.9$.
- Suppose 5 is sufficiently large.
  - Look up $z_{0.95} = 1.645$ for one-sided critical value.
  - $\mathbf{Pr}\,(0.1040 \leq \mu \leq 0.1136) = 0.9$.
  - But you know, 5 will not be sufficiently large.
- Take Student's distribution.
  - Look up $t_{4,0.95} = 2.132$.
  - $\mathbf{Pr}\,(0.1026 \leq \mu \leq 0.1150) = 0.9$.

ANSWERING A YES/NO QUESTION

- No concrete estimation of a value possible.
- Reject or accept educated guess, but
  - there is no estimation of the real value.
- The question is different from value estimation.
- We want to know if we can accept/reject a particular assertion, usually called null-hypothesis.

Hypothesis testing

- Formulate two mutually exclusive and exhaustive hypotheses,
  - null-hypothesis H_0, here $f_0(x)$,
  - alternative hypothesis H_1, here $f_1(x)$.
- Run a simulation and depending on the results
  - accept H_0,
  - accept H_1.
- Since we are only taking samples, errors are involved
  - type I-error, significance, α error,
  - type II-error, β.
- α is called the wrong negative,
  - the probability to reject H_0 although it is true.
- β is called the wrong positive,
  - the probability to accept H_0 although H_1 is true.

SEQUENTIAL SAMPLING

- Instead of having a fixed sample size
  - evaluate probabilities after each sample.
- During simulation it could be that
  - there is enough evidence to
    - accept H_0,
    - reject H_0,
  - there is no evidence to accept/reject H_0.
- A fixed sample size ignores this.
- With sequential sampling: decide after each sample if it is either
  - true, or
  - false, or
  - another sample is required.

Developing a test

- Sample space $M_m, m = 1, 2, \ldots, \infty$,
  - M support of probability distribution,
  - $a \in M_m$ is called sample point of size m.
- Divide M_m in
  - R_m^0,
  - R_m^1,
  - R_m.
- Termination: sample point of size n falls in
  - R_n^0, accept H_0,
  - R_n^1, accept H_1.

Computing probabilities

- k samples have been taken.
- $g_{i,k}$: probability that H_i is true after k observations.
- $p_{i,k}$: probability density in $k-$dimensional sample space, assuming H_i is valid.
- $g_{i,k} := \frac{p_{i,k}(x_1,x_2,\ldots,x_k)}{\sum_i p_{i,k}(x_1,x_2,\ldots,x_k)}$.
- Accept either of the hypothesis if $g_{i,k}$ is above d_i.
- If
  - $\frac{p_{1,k}(x_1,x_2,\ldots,x_k)}{p_{0,k}(x_1,x_2,\ldots,x_k)} \geq A$, accept H_1,
  - $\frac{p_{1,k}(x_1,x_2,\ldots,x_k)}{p_{0,k}(x_1,x_2,\ldots,x_k)} \leq B$, accept H_0.
- It can be shown
  - $A \approx \frac{1-\beta}{\alpha}$,
  - $B \approx \frac{\beta}{1-\alpha}$.

PRECISELY

- $A := \frac{1-\beta}{\alpha}$.
- $B := \frac{\beta}{1-\alpha}$.
- $\lambda_n := \frac{p_{1,k}(x_1,x_2,\dots,x_k)}{p_{0,k}(x_1,x_2,\dots,x_k)}$.
- Continue sampling when
    - $B < \lambda_n < A$.
- Accept H_0 when
    - $\lambda_n \leq B$.
- Reject H_0 when
    - $\lambda_n \geq A$.

Airbag deployment example - 1

- FoVW car company's airbag deployment rate.
- Required rate 98%.
- Recent customer reports indicate rate 80%.
- Define H_0.
  - $f_0(x) = \begin{cases} 0.80 & , x = 1, \\ 0.20 & , x = 0. \end{cases}$
- Define H_1.
  - $f_1(x) = \begin{cases} 0.98 & , x = 1, \\ 0.02 & , x = 0. \end{cases}$
- Choosing the errors.
  - $\alpha = 0.01$ (false negative),
  - $\beta = 0.05$ (false positive).

Airbag deployment example - 2

Boundaries

- $A := \frac{1-\beta}{\alpha} = 95.$
- $B := \frac{\beta}{1-\alpha} = 0.051.$

Taking samples

1. Airbag deploys.
   - $0.051 < \lambda_1 = \frac{f_1(1)}{f_0(1)} = \frac{0.98}{0.80} = 1.225 < 95.$
2. Airbag deploys not.
   - $0.051 < \lambda_2 = \frac{f_1(1,0)}{f_0(1,0)} = \frac{0.98 \cdot 0.02}{0.80 \cdot 0.20} = 0.1225 < 95.$
3. Airbag deploys not.
   - $\lambda_3 = \frac{f_1(1,0,0)}{f_0(1,0,0)} = \frac{0.98 \cdot 0.02^2}{0.80 \cdot 0.20^2} = 0.01225 < 0.051.$
4. Conclusion: accept H_0.

USING DES FOR CSL MODEL-CHECKING

- Checking $\mathcal{P}_{\geq\theta}(\rho)$,
  - ρ is a CSL path formula.
- Simple case. No nesting of probabilistic statements.
  - ρ does not contain $\mathcal{P}$.
  - Truth value of ρ can be determined without error.
  - Sequential sampling can be applied directly,
  - choose boundaries carefully.
- Complicated case. Nesting of probabilistic statements.
  - ρ contains $\mathcal{P}$ operator.
  - Truth value of ρ can be erroneous.
  - How to handle innermost formula?
  - Adjust error bounds α' and β' for inner most formula,
  - could lead to more samples!

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